Basic Probability and Statistics Review

Six Sigma Black Belt Primer

Pat Hammett, Ph.D.

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Instructor Comments:

This document contains a review of basic probability and statistics. It also includes a practice test at the end of the document. Note: answers to the practice test questions are included in an appendix.

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1. VARIABLES– QUALITATIVE AND QUANTITATIVE

A variable is any measured characteristic or attribute that differs for different subjects. For example, if the length of 30 desks were measured, then length would be a variable.

Key Learning Skills –
- Understand the difference between a qualitative (categorical) variable and a quantitative variable.
- Understand the types of qualitative (categorical) variables: Nominal, Ordinal, and Binary.
- Understand the difference between a discrete and a continuous quantitative variable.

Terms and Definitions:

1.1   Qualitative Data (Categorical Variables or Attributes)

Qualitative data involves assigning non-numerical items into groups or categories. Qualitative data also are referred to as categorical data. The qualitative characteristic or classification group of an item is an **attribute**. Some examples of qualitative data are:
- The pizza was delivered on time.
  - Categorical Variable: Delivery Result
  - Attribute: On Time, Not On Time
- The survey responses include disagree, neutral, or agree.
  - Categorical Variable: Survey Response
  - Attribute: Disagree, Neutral, Agree
- This car comes in black, white, red, blue, or yellow.
  - Categorical Variable: Color
  - Attribute: Black, White, Red, Blue, or Yellow.

Categorical variables are typically assigned attributes using a nominal, ordinal, or binary scale.
- **Nominal** variables are categorical variables that have three or more possible levels with no natural ordering. Car color would be considered a nominal variable. Again, in a nominal scale, no

quantitative information is conveyed and no ordering of the items is implied. Other examples of nominal scales include religious preference, production facility, and organizational function.

- **Ordinal** variables are categorical variables that have three or more possible levels with a natural ordering, such as strongly disagree, disagree, neutral, agree, and strongly agree. With ordinal data, quality analysts often convert it to a quantitative scale. For example, a survey may assign a scale from 1-5 to cover the range from strongly disagree, to neutral, to strongly agree. When converting an ordinal categorical variable to a quantitative scale, a quality analyst must exercise caution in the interpretation of the difference between values. For instance, the difference between the responses strongly disagree (1) and disagree (2) may not equal the difference between disagree (2) and neutral (3).

- **Binary** variables are categorical variables that have two possible levels (e.g., yes/no). Binary variables are the most common type of categorical variables because they are the easiest to convert to a quantitative scale. Binary variables typically are assigned a 0 (e.g., defective) or 1 (e.g., not defective). This use of the 0 / 1 designation allows experimenters to use proportions or counts for data analysis. As a general rule, the desired outcome is assigned the 1.

1.2   Quantitative Data

Quantitative Data result from measurement or numerical estimation. These measurements yield **discrete or continuous variables**. Discrete variables vary only by whole numbers such as the number of students in a class (variable: class size). Continuous variables vary to any degree, limited only by the precision of the measurement system. Some examples include the width of a desk, the time to complete a task, or the height of students (variables: length, time, and height). In the case of measuring the width of a desk, the measurement could read 1.54 m, or 1.541 m, or 1.5409, or 1.54087, ... Here, the observed measurement is limited only by the precision of the measurement instrument.

Some additional examples of continuous quantitative measurements are:
- The time to deliver the pizza was 26.7 minutes.
- The diameter of the cylinder was 83.1 mm.

In converting a categorical variable to a quantitative scale, the variable is typically treated as a discrete variable. For example, a rating scale from 1 to 5 or a binary scale of 0 or 1 would be analyzed as a discrete variable. In computing a statistic for a discrete variable such as the average survey response, the statistic (e.g., the average) is considered continuous. So, the average for a 5-points scale might be 3.72 even though this particular value is not possible to obtain.

For analysis purposes, discrete variables often are approximated using continuous distributions. For instance, suppose student test scores are discrete ranging from 0 to 100 points. Here, we might assume the distribution of test scores follows a normal distribution (continuous) in order to estimate the likelihood of a student scoring greater than a 70.

In general, analysts try to convert all data to an approximately continuous, numerical scale for making inferences or conclusions.

2. DESCRIPTIVE STATISTICS

Descriptive statistics are used to summarize the characteristics of a data set.

Key Learning Skills –
- Understand the difference between a sample and a population.
- Understand the difference between a parameter and a statistic.
- Compute the mean, median, standard deviation, variance and the range for a sample data set.

Terms and Definitions:

2.1 Sample Data versus Population Data

A population data set includes all items of the set, such as the height of every person in the United States, or the volume of every can of soda pop that a manufacturer produces. If the desired information is available for all items in the population, we have what is referred to as a **census**. In practice, we rarely have a complete set of data. We usually collect data in samples, such as the volumes of the last thirty cans of pop.

2.2 Parameters and Statistics

Numbers used to describe a population are *parameters* and often are denoted using Greek letters. Numbers used to describe a sample data set are called *statistics*. A Statistic may be used to estimate a population parameter such as the average of a data set ($\overline{X}$ *or* $\hat{m}$) provides an estimate of the population mean, μ.

The difference between a statistic and a parameter is important to understand because in statistical data analysis we often make *inferences* about a population based on sample statistics. Since we rarely know every observation in a population, any conclusions or recommendations that are made based on sample statistics are subject to error. However, we typically will accept some margin of error rather than incur the cost of measuring every observation.

2.3 Location Statistics (measures of central tendency)

Mean (also known as the average) is a measure of the center of a distribution.

$$Mean = \frac{X_1 + X_2 + ...X_N}{N}$$ Note : Mean is also referred to as $\overline{X}$

The typical notation used to represent the mean of a sample of data is $\overline{X}$; the Greek letter μ is used to represent the mean of a population. The terms, $\overline{X}$ *or* $\hat{m}$, represent the estimate of the population mean.

Example: suppose five students take a test and their scores are 70, 68, 71, 69 and 98.

Mean = (70+68+71+69+98)/5 = 75.2

Notice: the mean may be strongly influenced by extreme values. If we excluded the student whose score was a 98, the mean would change to 69.5.

Median (also known as the 50th percentile) is the middle observation in a data set. To determine the median, we rank the data set and then select the middle value. If the data set has an odd number of observations, the middle value is the observation number [N + 1] / 2. If the data set has an even number of observations, the middle value is extrapolated as midway between observation numbers N / 2 and [N / 2] + 1.

In the above example, the data ranked is 68, 69, 70, 71, and 98. Here, the median is 70. If another student with a score of 60 was included, the new median would 69.5 (69 + 70 / 2).

The median is often used if the data has extreme values (outliers) or is skewed (e.g., if one of the tails of a bell-shaped curve is significantly longer than the other). In the above example of student test scores, the median provides a better representation of the center of the distribution since 98 is an extreme value.

2.4 Dispersion Statistics (measures of variability)

Standard deviation (StDev) measures the dispersion of the individual observations from the mean. In a sample data set, the standard deviation is also referred to as the sample standard deviation or the root-mean-square S_{rms} and may be calculated using the following formula.

$$S = \sqrt{\frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{n-1}}$$

Note: to compute the population standard deviation, we use the population mean and divide by n instead of n-1. In practice, the population standard deviation is rarely used because the true population mean is usually unknown. The use of the sample standard deviation is particularly important for smaller sample sizes. However, as the sample size gets large (say n > 100), the difference between dividing by n versus n-1 may become negligible.

The typical notation used to represent the sample standard deviation is S; the Greek letter σ is used to represent the population standard deviation. The terms, S or $\hat{s}$, represent the estimate of the population standard deviation.

In the example with the five student test scores (70, 68, 71, 69 and 98), the sample standard deviation is 12.79.

Similar to the mean, the standard deviation may be strongly influenced by extreme values. If we exclude the student whose score was a 98, the sample standard deviation would be reduced to 1.3!

Variance is sometimes used to represent dispersion. The variance is simply the standard deviation squared. The variance represents the average squared deviation from the mean.

$$S^2 = \frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{n-1}$$

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Above Example: Variance = $(12.79)^2$ = 163.72

Note: the variance is often used because of its additive property. If several independent factors contribute to the overall variance, then the total variance may be determined by adding the individual factor variances (assuming the factors are independent). ***Note: we do not add standard deviations!***

Range is another measure of dispersion. The range is simply the maximum value in a data set minus the minimum value. In the above example, the range of (70, 68, 71, 69 and 98) is (98 - 68 = 30).

Note: the range is sometime preferred over the standard deviation to represent dispersion for small data sets (e.g., # of samples < 10).

3. FREQUENCY DISTRIBUTIONS

Frequency is used to describe the number of times a value or a range of values occurs in a data set. Cumulative frequencies are used to describe the number of observations less than, or greater than a specific value.

Key Learning Skills –
- Understand the difference between absolute, relative, and cumulative frequencies.
- Generate a frequency table.
- Generate a histogram.

Terms and Definitions:

3.1 Frequency Measures

Absolute frequency is the number of times a value or range of values occurs in a data set. The *relative frequency* is found by dividing the absolute frequency by the total number of observations (n). The *cumulative frequency* is the successive sums of absolute frequencies. The *cumulative relative frequency* is the successive sum of cumulative frequencies divided by the total number of observations.

To demonstrate the differences between these terms, consider the results of throwing a pair of dice. The possible combinations and their sums are shown in the following frequency table. Four measures of frequency are shown: absolute frequency, cumulative frequency, relative frequency, and cumulative relative frequency.

Frequency Table

Combination	Sum	Absolute Frequency	Cumulative Frequency	Relative Freq	Cum. Rel. Freq,
(1,1)	2	1	1	0.03	0.03
(1,2) (2,1)	3	2	3	0.06	0.08
(1,3) (3,1) (2,2)	4	3	6	0.08	0.17
(1,4) (4,1) (2,3) (3,2)	5	4	10	0.11	0.28
(1,5) (5,1) (2,4) (4,2) (3,3)	6	5	15	0.14	0.42
(1,6) (6,1) (2,5) (5,2) (3,4) (4,3)	7	6	21	0.17	0.58
(2,6) (6,2) (3,5) (5,3) (4,4)	8	5	26	0.14	0.72
(3,6) (6,3) (4,5) (5,4)	9	4	30	0.11	0.83
(4,6) (6,4) (5,5)	10	3	33	0.08	0.92
(5,6) (6,5)	11	2	35	0.06	0.97
(6,6)	12	1	36	0.03	1.00
	Total	36			

3.2 Histogram

A histogram is a graphical representation of a frequency table. Histograms also are used to show the shape of a distribution. Some common shapes are bell-shaped (i.e., normal), exponential or skewed. Skewed distributions are similar to normal distributions only one tail is significantly larger than the other. For example, a skewed right distribution has a basic bell-shaped curve with a longer tail on the right (or on the left). The figure below shows each of these shapes.

| Normal	Skewed Right

Exponential (Decreasing)
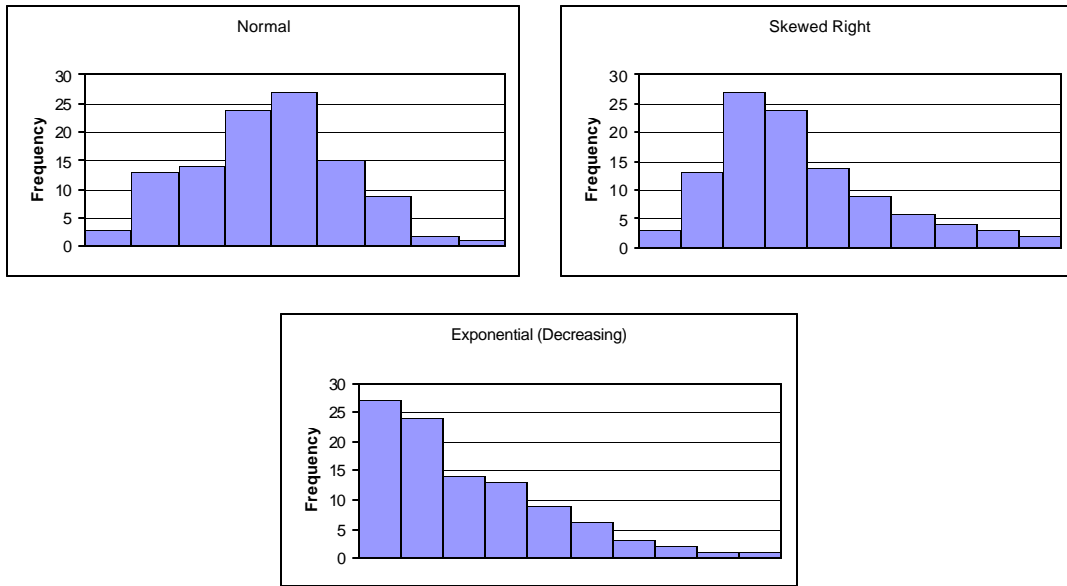
Some Example Histogram Shapes

In a histogram, each column represents the absolute frequency or relative frequency for a particular combination or occurrences in a data set of a single variable. Histograms may be used for discrete or continuous variables.

3.3 Discrete Histogram
Example: Possible Combinations for Sum of Pair of Dice
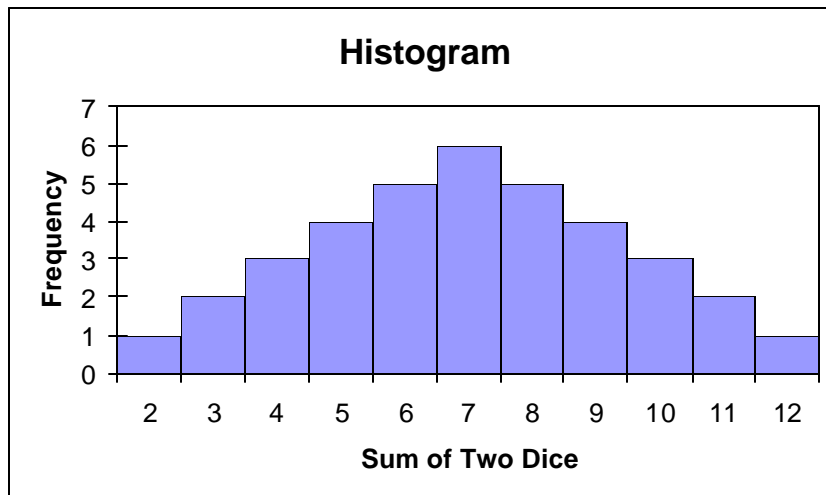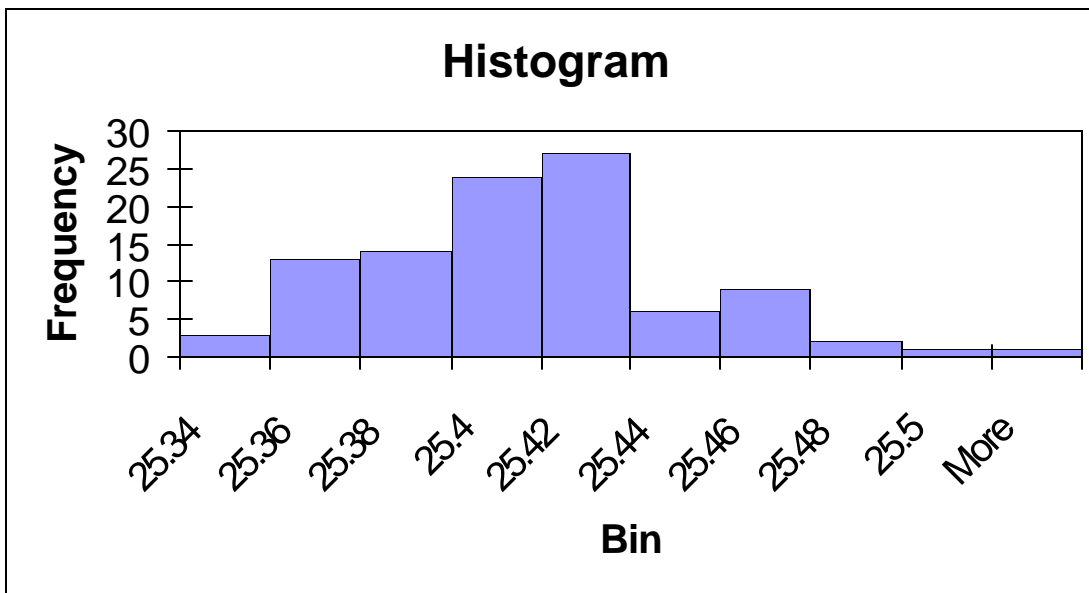

Histogram

3.4 Continuous Data Histogram

To demonstrate a continuous data histogram, suppose you obtain the following data set by measuring the diameter of 100 bicycle seat posts.

25.36	25.34	25.39	25.45	25.37
25.36	25.40	25.35	25.36	25.37
25.40	25.41	25.41	25.35	25.38
25.39	25.39	25.37	25.44	25.42
25.40	25.36	25.37	25.39	25.39
25.40	25.38	25.33	25.36	25.43
25.42	25.41	25.41	25.37	25.40
25.47	25.41	25.32	25.46	25.40
25.39	25.42	25.41	25.42	25.35
25.42	25.41	25.42	25.41	25.46
25.40	25.40	25.43	25.36	25.41
25.44	25.46	25.41	25.37	25.36
25.38	25.50	25.38	25.40	25.40
25.40	25.39	25.36	25.36	25.44
25.38	25.38	25.39	25.40	25.35
25.41	25.34	25.39	25.40	25.34
25.55	25.43	25.42	25.41	25.39
25.40	25.36	25.42	25.41	25.45
25.45	25.39	25.40	25.36	25.41
25.35	25.43	25.40	25.38	25.38

First, arrange the data into frequency or bin ranges of equal width. The selection of the number and width of the bins (frequency ranges) is dependent on the analyst. For continuous data, a general rule of thumb is to set the number of bins equal to the square root of the number of samples (rounded to nearest whole number). To obtain the bin width, divide the range of the data set by the number of bins (rounded to desired precision of measurement data). This example has 100 samples and a range of 0.23. Thus, an analyst might create 10 bins (=sqrt(100)) of width 0.02 mm (0.23/10 = 0.02). In this example, the value 25.34 was chosen as the starting point because relatively few values are below it.

Bin Range	Absolute Frequency	Cumulative Frequency
< 25.34	3	3
25.34 < X <= 25.36	13	16
25.36 < X <= 25.38	14	30
25.38 < X <= 25.4	24	54
25.40 < X <= 25.42	27	81
25.42 < X <= 25.44	6	87
25.44 < X <= 25.46	9	96
25.46 < X <= 25.48	2	98
25.48 < X <= 25.5	1	99
X > 25.50	1	100

Interpreting a Continuous Data Histogram – the absolute frequencies in a continuous data histogram represent the number of observations that fall within a range. In the graph above, the first column (labeled 25.34) represents the number of observations less than or equal to 25.34. The second column (labeled 25.36) represents the number of observations greater than 25.34 and less than or equal to 25.36. The third column is the number of observations greater than 25.36 and less than or equal to 25.38.

4. PROBABILITY AND ERROR

Probability describes the likelihood that an event will occur. An event may represent an outcome such as the probability of rolling a three with a dice, or it could represent a recommendation or decision from an analysis, such as two machines produce different outputs. In this case, probability is used to convey confidence or risk in the result or conclusion.

Key Learning Skills –
- Understand difference between a priori probability and empirical probability.
- Understand basic probability concepts and rules.
- Understand type I and type II errors.

Terms and Definitions:

4.1 Probability Properties

Probability is the likelihood that an event will occur. Probabilities are expressed as values between 0 and 1 where 0 is the null probability and 1 is certainty.

In a probability assessment, the event may be either desired or not desired. For instance, we might be interested in the probability that a part, measured at random, is within its specification limits, or *not* within its specification limits.

Methods Used to Calculate Probability

- **A Priori Probability** – may be derived before data are collected. (Often used if data are discrete and the possible outcomes are finite, such as in games of chance.)

- **Empirical Probability** – is estimated from a sample data set. (Often used in industrial and experimental situations where data are continuous and the underlying nature of events is unknown.)

A Priori Probability may be obtained if all possible events and outcomes may be enumerated or mathematically described. The a priori probability may be determined using the following formula:

P = E / N
Where:
P = probability
E = number of cases in which a desired event or outcome occurs
N = total number of possible cases.

In the dice example from the prior section, the probability a person will roll a seven using a pair of dice is.

P = 6 / 36 = 0.17 (Note: (1,6) (6,1) (2,5) (5,2) (3,4) (4,3) are six out of thirty-six possible combinations whose sum equals seven.)

Similarly, the probability a person will roll a seven or less using a pair of dice is:

P = 21 /36 =0.58

Empirical Probability – In most cases, either the number of times an event might occur or the total number of possible outcomes is unknown. For example, if data are sampled from a continuous distribution such as measuring the height of twelve-year old boys, both the possible number of occurrences of each height and the total number of possible outcomes are unknown. Here, the probability of an event, such as the likelihood that a twelve year-old boy is taller than 60 inches must be derived empirically based on sampling.

A common method for empirically estimating probabilities is to sample data and then fit the data into a statistical distribution. Given a statistical distribution, the probability of an event may be derived using estimates of the distribution parameters. The most widely used distribution for estimating probabilities is the Normal Distribution. (See Section on Normal Distribution.)

4.2 Type I and II Errors

Probability is often used to convey confidence or assess risk when making a decision or recommendation from a statistical analysis. For example, a company may decide to inspect 100% of their products if they find a defect in a random sample of ten finished products. Or, they may decide to re-calibrate a machine if they determine from a sampling of products across several machines that a particular machine is producing a different mean.

In performing a statistical analysis, an experimenter must begin with a claim or statistical hypothesis. This claim or hypothesis is what you assume to be true. In assessing claim, four possible outcomes may result:
   a. Conclude the claim is true, when it actually is true
   b. Conclude the claim is false, when it actually is true.
   c. Conclude the claim is true, when it actually is false.
   d. Conclude the claim is false, when it actually is false.

Of the above outcomes, two of them (b and c) are errors. These errors are referred to as Type I (alpha) or Type II (beta) errors.

- Type I error (alpha error) – occurs when one concludes something is not true (e.g., say different) when in fact it is true (truth = not different). For example, a Type I error might occur if the particular parts sampled are not representative of the distribution such as if a part is not measured correctly. Here, an analysis might show to adjust machine settings to correct a perceived problem when in fact the machine is working properly.

- Type II error (Beta risk) – occurs when one concludes something is true (e.g., say not different) when in fact it is false (i.e., different).

A jury trial offers a good analogy for comparing Type I and II errors. In a jury trial, four possible outcomes may occur: an innocent person may be found innocent, an innocent person may be found guilty, a guilty person may be found innocent, and a guilty person may be found guilty. These outcomes and their associated statistical terms are shown below.

		Truth	
		Innocent (true or not different)	Guilty (false or different)
Say or Conclude	Innocent (true or not different)	Correct Conclusion $(1 - \alpha)$	Type II error (beta risk– β)
	Guilty (false or different)	Type I error (alpha – α)	Correct Conclusion $(1 - \beta$ = Power$)$

The probability of making a Type I error is controlled by establishing a probability threshold known as the alpha error. Some typical levels of alpha are 0.05 or 0.01. Here, the confidence of making the right decision is 1 – alpha. For these alpha values, we may want to be 95% or 99% confident in our decision. In other words, we may want to be at least 95% confident that if we accept a claim as true, that it is true.

The probability of making a Type II error is usually controlled by the selection of an appropriate sample size. In general, the larger the sample size, the greater the ability to detect a difference. Thus, increasing sample size results in greater power in making a correct decision. The probability of finding a difference (e.g., say different) when a difference actually exists is known as statistical power (1-β). In other words, we may want to have at least 90% power that if we suggest a claim is false, that it is false.

Jury Trial Error Analogy

As with jury trials, the willingness to accept risk (alpha error) is highly dependent on the significance of the outcome. For example, in the U.S. legal system, a greater concern exists for Type I errors in high crimes. The U.S. system would rather conclude the innocence of a guilty person, than convict people that are innocent. To reduce the Type I error in a high crime, U.S. system requires a unanimous jury decision. In contrast, a review of the interpretation of a law by a panel of judges may only require a majority. Here, the legal system is willing to accept greater risk of a Type I error.

4.3 p-values and statistical significance

When conducting a statistical analysis, the p-value is used to represent the probability that no difference exists (for example, two machines are not producing statistically different mean outputs). A common method for determining significance in a statistical comparison is to conclude a difference exists if the p-value is less than the alpha error level.

For example, suppose you determine in a comparison of two bottle-filling machines that the probability that the two means are not different is 0.24. Assuming alpha = 0.05, you would conclude that the two means are not different. However, if the p-value is 0.003, you would conclude that the means *are* different. In this example, you are 99.7% confident that you are making the correct decision.
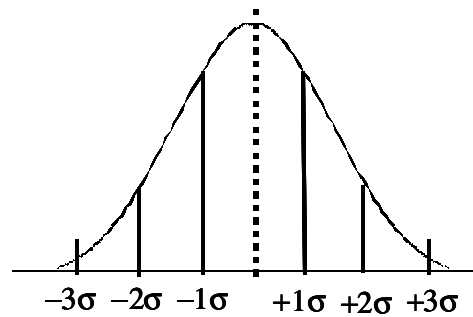
5. NORMAL DISTRIBUTION

The Normal distribution, visually resembling a smooth, symmetrical, bell-shaped curve, represents a commonly found pattern of randomly measured data. It is used to describe a great variety of situations such as intelligence test results, part measurements from automatic machines, measurement errors from a gage, etc. In fact, the failure to find a normal distribution when studying a continuous process often suggests that some factor is exerting an unusual amount of influence on the process (special cause of variation exists).

Key Learning Skills –
- Understand Some Common Properties of Normal Distribution and the Standard Normal Distribution.
- Estimate the probability of an event given that the observed data follow a normal distribution.

5.1 Properties of the Normal Distribution

The figure below shows a normal distribution. In a normal distribution, the mean, median, and mode all coincide. In addition, the number of standard deviations about the mean may be represented by probabilities. For example, if data are normally distributed, then 99.73% of values should fall between +/- 3σ.

-3σ -2σ -1σ $+1\sigma$ $+2\sigma$ $+3\sigma$

+/- 1σ = 68.26%
+/- 2σ = 95.46%
+/- 3σ = 99.73%

Properties of a Normal Distribution

5.2 Estimating Probabilities Using Normal Distribution

If data are normally distributed (or reasonably assumed to be normal), event probabilities may be empirically derived based on parameter estimates. To use the normal distribution, values are first converted to standardized Z scores. To standardize data, we use the following transformation:

$Z = (X - \mu) / \sigma$

Z scores transform data into the standard cumulative normal distribution whose mean = 0, and variance (σ^2) = 1. Z-scores provide a mapping from a distribution of some variable to a standardized scale. These mappings reflect the difference in terms of number of standard deviations away from the mean. If the mean of a process = 4 mm and the standard deviation = 1, then an observed value of 1 could also be represented as –3*standard deviation from the mean. For this example, a Z = -3 is equivalent to an actual observation of 1 (where Z = –3*standard deviation away from the mean).

By standardizing data, the probability of an event may be obtained by using the Z-scores. For example, suppose you wanted to compute the probability that a value falls between 4 and 16 given a mean = 10 and a standard deviation = 2.

Pr (4 < X < 16) = Pr (X < 16) – Pr (X < 4)

Z (X = 4) = (4 – 10) / 2 = -3.0
Z (X = 16) = (16 – 10) / 2 = 3.0

Pr (Z< -3.0) = 0.00135
(See appendix for Standardize Normal Curve Table)

Pr (Z< 3.0) = 0.99865 (or 1 – 0.00135)
(See appendix for Standardize Normal Curve Table)

Pr (Z < 3) – Pr (Z < -3) = 0.99865 – 0.00135 = 0.9973

Thus, 99.73% of values will fall between +/- 3σ.

5.3 Calculating Parts Per Million Defects Given Normal Distribution

If data are normally distributed, parts per million defects may be estimated using the standardized normal curve. The following process provides a step-by-step process for calculating parts per million defects assuming that the data follow a normal distribution and the process has a bilateral tolerance.

Step 1: Obtain necessary input data information.
- Specifications: Target, Upper Specification Limit (USL), and the Lower Specification Limit (LSL).
- Summary Statistics from Data Set: Estimate of the Sample Mean and Standard Deviation.

Example: Suppose you are trying to bicycle seats whose diameter specification is 25.4 +/- 0.05. You sample 100 parts and obtain a mean = 25.41 and sample standard deviation = 0.02.

Target = 25.4; USL = 25.45; LSL = 25.35; Mean = 25.41; Std Dev = 0.02

Step 2: Pictorially show the USL, LSL, Target, Mean, and Std Deviation

TIP: identify whether the mean is closer to the USL or the LSL as the defects per million should be greater on the side that is closest to the mean.

Example: graph of the above problem.

Mean = 25.41

LSL = 24.35

USL = 25.45

Target
25.4
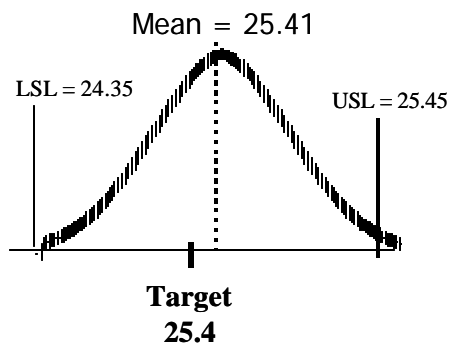
Step 3: Calculate the probability of a defect above the USL and below the LSL.

3a. Calculate Pr(Defect > USL). To obtain the probability that a part will be greater than the USL, we need to calculate a Z-value for the USL (Zusl) and look up the probability in a normal probability table. Note: we may also use an Excel built-in function to obtain this probability.

> Compute Zusl = (USL – Mean) / std deviation
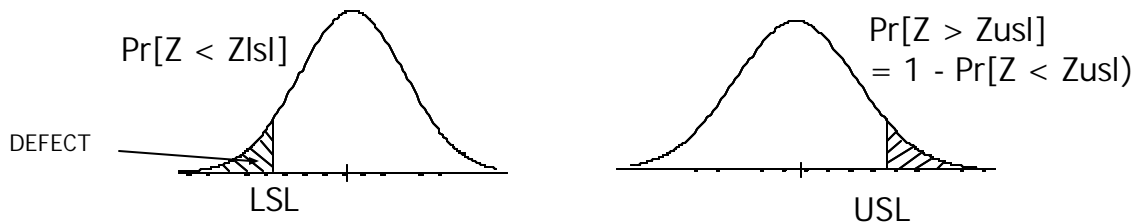> From Zusl, we may determine the Pr (Defect > USL).
> Pr (Defect > USL) = 1 – Pr(Z<Zusl).

Normal probability tables are presented as the probability from negative infinity to Z. Thus, for calculating defects greater than the USL, we need to let Pr (Defect > USL) = 1 – Pr (Z < Zusl). Pr(Z < Zusl) is obtained by looking up the value for Zusl in a normal probability table.

> Example:
> Target = 25.4; USL = 25.45; LSL = 25.35;
> Mean = 25.41; Std Dev = 0.02
>
> Zusl = (25.45 – 25.41) / 0.02 = 2.00
>
> Pr (Z < Zusl) = 0.97725 (based on Normal Table where Zusl = 2.0)
> Alternatively in Excel: =normsdist(2.0)   → 0.97725
>
> Pr (Defect > USL) = 1 – Pr (Z < Zusl) = 1 – 0.97725 = 0.02275

3b. Calculate Pr(Defect < LSL). To obtain the probability that a part will be less than the LSL, we need to calculate a Z-value for the LSL (Zlsl) and look up the probability in a normal probability table. Note: we may also use an Excel built-in function to obtain this probability.

> Compute Zlsl = (LSL – Mean) / std deviation
> From Zlsl, we may determine the Pr (Defect < LSL).
> Pr (Defect < LSL) = Pr(Z< Zlsl).

Normal probability tables are presented as the probability from negative infinity to Z. Thus, for calculating defects less than the LSL, we need to let Pr (Defect < LSL) = Pr (Z< Zlsl). Pr(Z < Zlsl) is obtained by looking up the value for Zlsl in a normal probability table.

> Example:
> Target = 25.4; USL = 25.45; LSL = 25.35;
> Mean = 25.41; Std Dev = 0.02
>
> Zlsl = (25.35 – 25.41) / 0.02 = -3.00
>
> Pr (Z < Zlsl) = 0.00135 (based on Normal Table Lookup where Z = -3.0)
> Alternatively in Excel: =normsdist(-3.0)   →   0.00135
>
> Pr (Defect < LSL) = 0.00135

Step 4: Calculate the probability of a defect.

> Pr (Defect) = Pr (Defect > USL) + Pr (Defect < LSL)
>
> Example: Pr (Defect) = 0.02275 + 0.00135 = 0.02410

Step 5: Calculate the Actual DPM

> Actual DPM = Pr (Defect) * 1,000,000
>
> Example: Actual DPM = 0.02410 * 1M = 24,100 DPM

Calculating Potential DPM:

We may want to calculate the Potential DPM, which represents the DPM that could be achieved if the process mean is shifted to the target value and the standard deviation does not change. To compute the potential DPM, repeat the above steps but substitute the target value for the mean. Note: Pr (Defect < LSL) should be equal to Pr (Defect > USL) if your target value is at the center of the USL and LSL. Also, your potential DPM should be less than your actual DPM if your current mean is not equal to your target value.

Example:
Zusl = (25.45 – 25.4) / 0.02 = 2.5 (table lookup → 0.99379)
Pr (Defect > USL) = 1- 0.99379 = 0.00621

Zlsl = (25.35 – 25.4) / 0.02 = -2.5 (table lookup → 0.00621)
Pr (Defect < LSL) = 0.00621

Pr (Defect) = 0.00621 + 0.00621 = 0.01242

DPM = 0.01242 * 1 M = 12,420 DPM

COMMENT: For this example, shifting the mean to the target value (given the same standard deviation) could potentially reduce the DPM by approximately one-half (24,100 to 12,420).

6. LINEAR REGRESSION ANALYSIS

Regression is used to describe relationships between variables.

Key Learning Skills –
- Compute the slope and y-intercept using a simple linear regression.
- Compute and interpret simple correlation coefficient.
- Understand the difference between simple linear regression and multiple linear regression.

Terms and Definitions:

6.1 General Regression equation

The regression equation is used to describe the relationship between the response variables and predictor(s). The general equation is:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \dots \beta_n X_n$$

Y – represents the response variable.
β_0 – represents the Y-intercept (value of response when predictor(s) variable = 0).
$\beta_{(1..n)}$ - is the slope or rate of change of each predictor variable.
$X_{(1..n)}$ - is the value of each predictor variable.

6.2 Simple linear regression

Simple linear regression examines the linear relationship between two variables: one response (y) and one predictor (x). If the two variables are related, the regression equation may be used to predict a response value given a predictor value with better than random chance. The simple linear regression equation is:

$$Y = \beta_0 + \beta_1 X_1$$

The most common method used to determine the line that "best" fits data is Least Squares Regression, which minimizes the squared deviations between individual observations and the regression line.

The equations used to compute the slope (β_1) and Y-intercept (βo) are:

$$b_1 = \frac{n\left(\sum (X_i Y_i)\right) - \left(\sum X_i\right)\left(\sum Y_i\right)}{n\left(\sum X_i^2\right) - \left(\sum X_i\right)^2}$$

$$bo = \frac{\sum Y_i - b_1\left(\sum X_i\right)}{n}$$

Note: n is the number of samples.

Alternatively, you could use the excel functions =slope(Y-array,X-array) and =intercept(Y-array,X-array). For example, if the Y-variable data are in Excel work cells B2:B10 and the X-variable data are in cells A2:A10, then the formula would be =slope(B2:B10,A2:A10).

Simple Linear Regression Example

Suppose you conduct an experiment to examine the relationships between bicycle tire pressure, tire width, and the coefficient of rolling friction. From experiments, you obtain the following:

Coefficient of Rolling Friction for Bicycle Tires

Pressure (PSI)	Width=1.25"	Width= 2"
20	0.0100	0.0107
25	0.0095	0.0100
30	0.0088	0.0093
35	0.0081	0.0086
40	0.0074	0.0079
45	0.0067	0.0073
50	0.0060	0.0071
55	0.0058	0.0068
60	0.0056	0.0066
65	0.0054	0.0063
70	0.0052	0.0061
75	0.0050	0.0058

Given these data, estimate the slope and Y-intercept for both variables.

Using excel, the following results may be obtained.

	Width=1.25"	Width= 2"
slope	-0.0001	-0.0001
intercept	0.0115	0.0118

6.3 Correlation

The Pearson correlation coefficient measures the extent to which two continuous variables are linearly related. For example, you may want to measure the correlation between tire pressure and the coefficient of rolling friction in the above example.

Simple correlation may be measured using the following equation:

$$R = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{(n-1)s_x s_y}$$

Using excel, =correl(Yarray,Xarray)

The correlation coefficient, R, consists of a value between –1 and 1. Perfect correlation (either –1 or 1) occurs when every observation in a sample falls exactly on the predicted line (i.e., no error).

Strong positive correlation (value approaching 1) exists when both variables increase or decrease concurrently. A correlation value, R, which is greater than 0.7, typically indicates a strong positive relationship.

Strong negative correlation (value approaching –1) exists if one variable increases while the other variable decreases. A correlation value, R, which is less than -0.7, typically indicates a strong negative relationship.

6.4 Using Scatter Plots to Show Linear Relationships

A scatter plot is an effective tool for viewing the strength (strong – weak correlation) and direction (positive and negative). The figures below show several examples with different correlation coefficients.
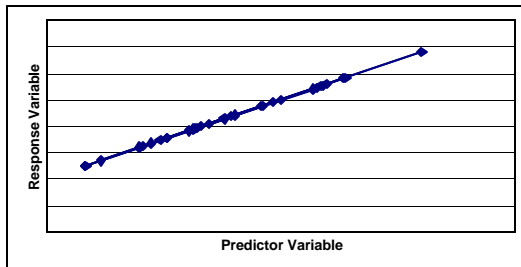
a) Perfect Positive Correlation (R = 1.0)
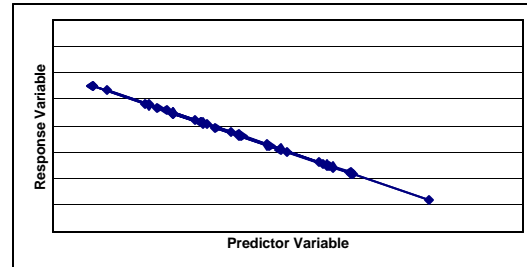
b) Perfect Negative Correlation (R = -1.0)
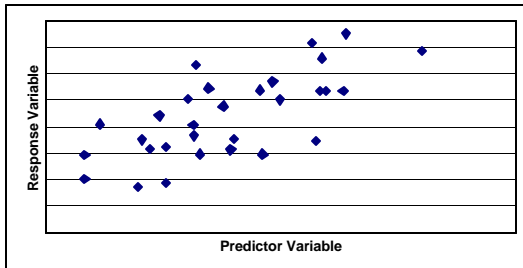
c) Strong Positive Correlation (R = 0.7)
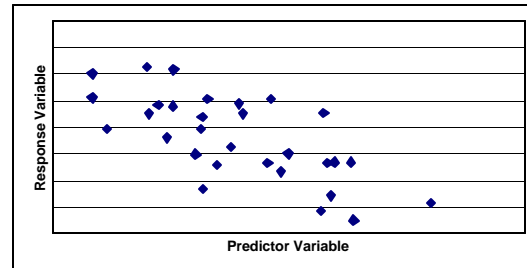
d) Strong Negative Correlation (R = -0.7)

Note: If two variables are normally distributed with no correlation (i.e., R=0), the resulting figure will resemble a circle.

Interpreting Correlation Coefficients

When drawing conclusions based on correlation coefficients, several important items must be considered:

- Correlation coefficients only measure linear relationships. A meaningful nonlinear relationship can exist even if the correlation coefficient is 0.

- Correlation does **NOT** always indicate cause and effect. One should not conclude that changes to one variable cause changes in another. Properly controlled experiments are needed to verify that a correlation relationship indicates causation.

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- A correlation coefficient is very sensitive to extreme values. A single value that is very different from the others in a data set can change the value of the coefficient a great deal. In the example below, the correlation is 0.9, but the scatter plot suggests that an outlier more likely explains the relationship that the predictor variable. If you removed the outlier value, the correlation between these two variable would drop to 0.1 over the smaller range of X.

Tonnage Vs. Drawdepth
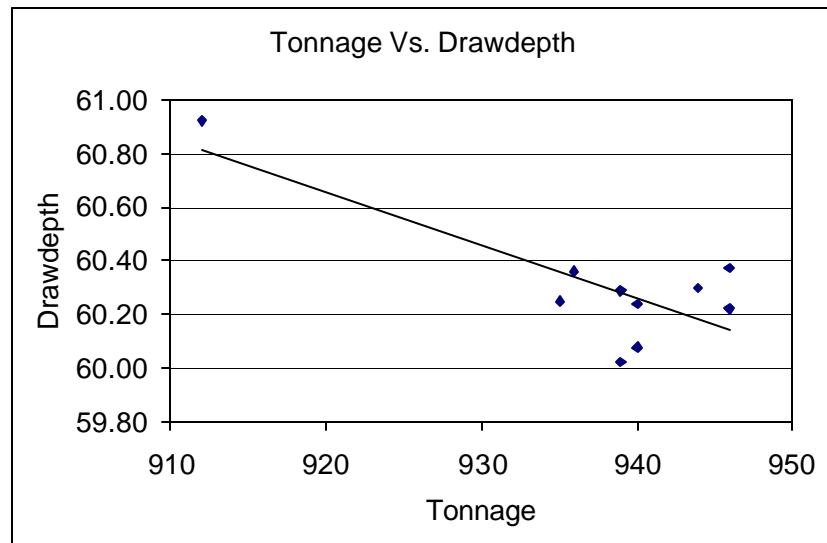
6.5 Multiple linear regression

Multiple linear regression examines the linear relationships between one continuous response and two or more predictors. If the response and predictor variables are related, the regression equation may be used to predict a response value given predictor values with better than random chance.

When using multiple linear regression, one should exercise caution if the number of predictors is large, particularly in relation to the sample size. For example, trying to fit a multiple regression model with 5 predictor variables and only 10 data points is likely to yield problems, especially if the predictor variables are not independent of each other (i.e., no relationship). Here, one should reference a more advanced regression technique.

A. PRACTICE TEST

In making calculations, feel free to use a calculator or software (Excel or Minitab). The answers also are available in an appendix. Note: these problems are based on this probability and statistics review manual.

Descriptive Statistics

Use this information from UM's 11-game football season to answer the following:

University of Michigan - Football Statistics by Game (Year 2000)

Game	UM-Score	Opp-Score	Point Difference	UM Offense	Opponent Offense
1	42	7	35	554	271
2	38	7	31	396	271
3	20	23	-3	374	394
4	35	31	4	513	447
5	13	10	3	375	278
6	31	32	-1	430	530
7	58	0	58	562	190
8	14	0	14	326	355
9	51	54	-3	535	654
10	33	11	22	444	407
11	38	26	12	389	400
N	11	11	11	11	11
Sum	373	201	172	4898	4197
Minimum	13	0	-3	326	190
Maximum	58	54	58	562	654

1. What is the average number offense per game for UM football team?

2. What is the standard deviation of offense per game for UM football team?

3. What is the median point difference (UM Score – Opponent Score)?

Frequency and Probability

Suppose that you roll a pair of dice. The following table summarizes the 36 possible combinations for their sum. Use this table to answer the following questions.

Combination	Sum	Absolute Frequency	Cumulative Frequency
(1,1)	2	1	1
(1,2) (2,1)	3	2	3
(1,3) (3,1) (2,2)	4	3	6
(1,4) (4,1) (2,3) (3,2)	5	4	10
(1,5) (5,1) (2,4) (4,2) (3,3)	6	5	15
(1,6) (6,1) (2,5) (5,2) (3,4) (4,3)	7	6	21
(2,6) (6,2) (3,5) (5,3) (4,4)	8	5	26
(3,6) (6,3) (4,5) (5,4)	9	4	30
(4,6) (6,4) (5,5)	10	3	33
(5,6) (6,5)	11	2	35
(6,6)	12	1	36
	Total	36	

4. What is the probability of rolling a seven (i.e., total sum of two dice = 7)?

5. What is the probability of rolling a five or less?

6. What is the probability of rolling a 5 through 9 (5, 6, 7, 8, 9)?

7. What is the probability of rolling two consecutive sevens (two sets of rolls)?

Probability and Error

8. A jury finding an innocent person guilty of a crime is an example of which type of error?

    a) Type I error (alpha)
    b) Type II error (Beta)
    c) Type III error

Normal Distribution

An automotive body manufacturer collects data on the height of their dash panel. They record all measurements as deviation from nominal (thus, the target value = 0). Based on a sample of 50 vehicles, they obtain the following information: Mean = 0.30 mm and Standard Deviation = 0.20 mm. The specification for dash height is 0 +/- 1 mm.

9. What is the probability that a vehicle will have a height above the upper specification limit?

10. Predict the part per million defects for dash panel height.

11. What percentage of products should fall between +/- 2.0 sigma of the process mean (0.30 mm)?

Simple Linear Regression and Correlation Analysis:

Use the following scatter plots to answer the following questions and information from the Descriptive Statistics Table.
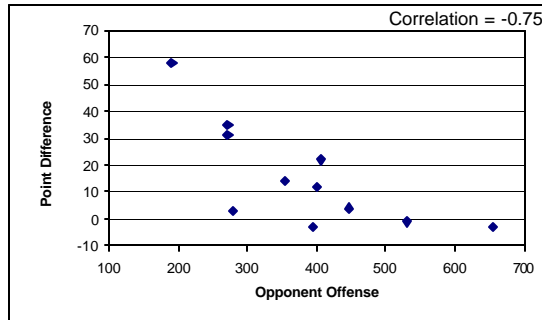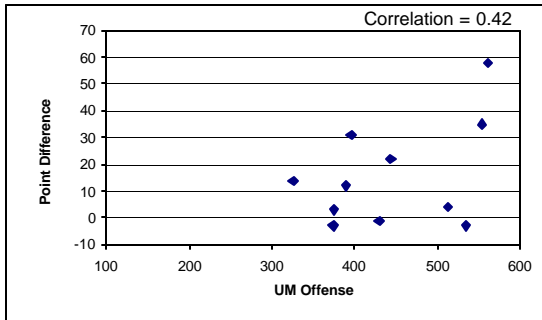
(UM Offense VS. Point Difference) (Opponent Offense Vs. Point Difference)

12. Which of the following has a strong negative correlation?

    a.   UM Offense and Point Difference
    b.   Opponent Offense and Point Difference
    c.   Both (a) and (b).
    d.   Neither (a) or (b).

13. Which of the following statements appears true based on the available data?

    a. The UM defense (measured by Opponents offense) is a better predictor of point difference than UM Offense.
    b. The UM offense is a better predictor of point difference than UM Defense.
    c. Cannot tell based on the information given.

14. What is slope of the best fit line between UM Offense and Point Difference?

15. What is the y-intercept of the best fit line between UM Offense and Point Difference?

Answers to Practice Test

1. Mean = <u>445.3</u>

2. StDev = <u>82.6</u>

3. Median = <u>12</u>

4. Pr (X = 7) = 6 / 36 = <u>0.167</u>

5. Pr (X <= 5) = 10 /36 = <u>0.278</u>

6. Pr (5 <= X <= 9) = 24 / 36 = <u>0.667</u>

7. Pr (Consecutive Sevens) = Pr (X = 7) * Pr (X = 7) = 0.167^2 = <u>0.028</u>

8. Say Guilty, truth = innocent → <u>Type I error (alpha)</u>

9. Pr (X > USL) = Pr [Z > (1-0.3)/0.2] → Pr (Z > 3.5) = <u>0.00023</u>

10. Pr (Z < -6.5) + Pr (Z > 3.5) x 1M = (~0 + 0.00023) x 1M = <u>230 DPM</u>

    Note : Zlsl = (-1 – 0.3) / 0.2 = - 6.5.

11. Pr (Z < 2) + Pr (Z < -2) = 0.97725 – 0.02275 = 0.9545 or <u>95.5%</u>

12. <u>(b) Opponent offense and point difference has strong negative correlation *R*</u>

13. <u>(a) UM defense has stronger correlation (-0.7) than UM offense (0.4)</u>

14. Slope (X – UM Offense, Y – Point Difference) = <u>0.098</u>

15. Y-Intercept (X – UM Offense, Y – Point Difference) = <u>-28.102</u>

Appendix B: Cumulative Normal Distribution Function (-Z values)
(Some examples: Z = -2.11 = 0.01743; Z = 1.50 = 0.93319)

Z	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.00
-3.5	0.00017	0.00017	0.00018	0.00019	0.00019	0.00020	0.00021	0.00022	0.00022	0.00023
-3.4	0.00024	0.00025	0.00026	0.00027	0.00028	0.00029	0.00030	0.00031	0.00032	0.00034
-3.3	0.00035	0.00036	0.00038	0.00039	0.00040	0.00042	0.00043	0.00045	0.00047	0.00048
-3.2	0.00050	0.00052	0.00054	0.00056	0.00058	0.00060	0.00062	0.00064	0.00066	0.00069
-3.1	0.00071	0.00074	0.00076	0.00079	0.00082	0.00084	0.00087	0.00090	0.00094	0.00097
-3.0	0.00100	0.00104	0.00107	0.00111	0.00114	0.00118	0.00122	0.00126	0.00131	0.00135
-2.9	0.00139	0.00144	0.00149	0.00154	0.00159	0.00164	0.00169	0.00175	0.00181	0.00187
-2.8	0.00193	0.00199	0.00205	0.00212	0.00219	0.00226	0.00233	0.00240	0.00248	0.00256
-2.7	0.00264	0.00272	0.00280	0.00289	0.00298	0.00307	0.00317	0.00326	0.00336	0.00347
-2.6	0.00357	0.00368	0.00379	0.00391	0.00402	0.00415	0.00427	0.00440	0.00453	0.00466
-2.5	0.00480	0.00494	0.00508	0.00523	0.00539	0.00554	0.00570	0.00587	0.00604	0.00621
-2.4	0.00639	0.00657	0.00676	0.00695	0.00714	0.00734	0.00755	0.00776	0.00798	0.00820
-2.3	0.00842	0.00866	0.00889	0.00914	0.00939	0.00964	0.00990	0.01017	0.01044	0.01072
-2.2	0.01101	0.01130	0.01160	0.01191	0.01222	0.01255	0.01287	0.01321	0.01355	0.01390
-2.1	0.01426	0.01463	0.01500	0.01539	0.01578	0.01618	0.01659	0.01700	0.01743	0.01786
-2.0	0.01831	0.01876	0.01923	0.01970	0.02018	0.02068	0.02118	0.02169	0.02222	0.02275
-1.9	0.02330	0.02385	0.02442	0.02500	0.02559	0.02619	0.02680	0.02743	0.02807	0.02872
-1.8	0.02938	0.03005	0.03074	0.03144	0.03216	0.03288	0.03362	0.03438	0.03515	0.03593
-1.7	0.03673	0.03754	0.03836	0.03920	0.04006	0.04093	0.04182	0.04272	0.04363	0.04457
-1.6	0.04551	0.04648	0.04746	0.04846	0.04947	0.05050	0.05155	0.05262	0.05370	0.05480
-1.5	0.05592	0.05705	0.05821	0.05938	0.06057	0.06178	0.06301	0.06426	0.06552	0.06681
-1.4	0.06811	0.06944	0.07078	0.07215	0.07353	0.07493	0.07636	0.07780	0.07927	0.08076
-1.3	0.08226	0.08379	0.08534	0.08692	0.08851	0.09012	0.09176	0.09342	0.09510	0.09680
-1.2	0.09853	0.10027	0.10204	0.10383	0.10565	0.10749	0.10935	0.11123	0.11314	0.11507
-1.1	0.11702	0.11900	0.12100	0.12302	0.12507	0.12714	0.12924	0.13136	0.13350	0.13567
-1.0	0.13786	0.14007	0.14231	0.14457	0.14686	0.14917	0.15151	0.15386	0.15625	0.15866
-0.9	0.16109	0.16354	0.16602	0.16853	0.17106	0.17361	0.17619	0.17879	0.18141	0.18406
-0.8	0.18673	0.18943	0.19215	0.19489	0.19766	0.20045	0.20327	0.20611	0.20897	0.21186
-0.7	0.21476	0.21770	0.22065	0.22363	0.22663	0.22965	0.23270	0.23576	0.23885	0.24196
-0.6	0.24510	0.24825	0.25143	0.25463	0.25785	0.26109	0.26435	0.26763	0.27093	0.27425
-0.5	0.27760	0.28096	0.28434	0.28774	0.29116	0.29460	0.29806	0.30153	0.30503	0.30854
-0.4	0.31207	0.31561	0.31918	0.32276	0.32636	0.32997	0.33360	0.33724	0.34090	0.34458
-0.3	0.34827	0.35197	0.35569	0.35942	0.36317	0.36693	0.37070	0.37448	0.37828	0.38209
-0.2	0.38591	0.38974	0.39358	0.39743	0.40129	0.40517	0.40905	0.41294	0.41683	0.42074
-0.1	0.42465	0.42858	0.43251	0.43644	0.44038	0.44433	0.44828	0.45224	0.45620	0.46017
0.0	0.46414	0.46812	0.47210	0.47608	0.48006	0.48405	0.48803	0.49202	0.49601	0.50000

Cumulative Normal Distribution Function (Positive Z-values)

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91308	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983

Appendix C: Some Useful excel statistical functions
(Note: an array represents a sample data set in excel)

Count   =count(array)
Sum   =sum(array)
Mean (average)   =average(array)
Median   =median(array)
Standard Deviation =stdev(array)
Variance   =var(array)
Maximum   =max(array)
Minimum   =min(array)

Percentile   =percentile(array,value)
                      e.g., to find the 95th percentile → =percentile(array,0.95)

Slope   =slope(Y array, X array)
Intercept   =intercept(Y array, X array)

Correlation   =correl(array 1, array 2)

Normal Distribution Function:
   =normdist(X,mean,standard deviation, true) → returns the Prob(x < X) using the
   cumulative normal distribution based on the specified mean and standard
   deviation.

   =normsdist(Z) → returns the Prob(z < Z) using the standard normal cumulative
   distribution (mean of zero and standard deviation of one).

   =normsinv(probability) → returns the inverse of the standard normal cumulative
   distribution for a given probability.

Random Number Generator – rand() may be used to generate a random number
between 0 and 1.